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Deriving the primary cumulative distribution function of fracture stress for brittle materials from 3- and 4-point bending tests

Constanze Przybilla [a], Alfonso Fernández-Canteli [a,*], Enrique Castillo [b]

[a] Department of Construction and Manufacturing Engineering, University of Oviedo, Campus de Viesques, 33203 Gijón, Spain
[b] Department of Applied Mathematics and Computational Sciences, University of Cantabria, Avda. de los Castros s/n, 39005 Santander, Spain

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Abstract

In this article, the primary three-parameter Weibull cumulative distribution function (cdf) of the critical stress provoking failure in a brittle material for a uni-axially and uniformly tensioned area ΔA is derived from 3- and 4-point bending test data. The model proposed finds application in the characterization of ceramics and glasses, and is intended as an initial step to be extended to different practical cases in future applications, as for instance, element design and local models in fracture mechanics, with previous consideration of the randomly distributed crack orientations. A comparison of the results provided by the model proposed with those found using another one referred to in the literature, demonstrates good agreement between both, whereas the former simplifies the convergence procedure and can be applied for the assessment of data obtained from different test geometries and types. Thus, the suitability of the new approach is confirmed.

Keywords: Failure analysis; Fracture; Mechanical properties; Weibull statistics

1. Introduction and motivation

As the strength of brittle materials exhibits high scatter, a statistical consideration of failure in form of a cumulative distribution function (cdf) should be used for its characterization. 3- and 4-point bending tests are widely used to determine this function experimentally. The resulting function clearly depends on the geometry of the tested samples. Thus, the so-called size effect has to be taken into account when the experimentally obtained function is used to calculate the probability of failure of a mechanical or structural component of much greater dimensions compared to the ones of the samples used in experiments and presenting local variation of the stress state. Furthermore, the cdf obtained should be related to an area subjected to uniform stress as this function is subsequently used in finite element calculations assuming the stress state in one cell to be constant. Thus, a relationship has to be established between the cdf of failure resulting for the variable stress distribution of 3- or 4-point bending tests currently used and that for a reference area under constant stress. This reference area, related to the dimensions and the stress state of the sample tested, is deduced under the assumption that the material strength can be described by a three-parameter Weibull distribution function which is widely used to evaluate failure test data from bending tests. As the herein derived reference area depends on the stress level associated to failure the three-parameter Weibull cdf cannot be directly determined from the test results by a simple change of the scale parameter. Instead, the reference area has to be considered for each data point individually to finally determine the cdf of a uni-axially tensioned area ΔA.

In foregoing work[1] a reference area has been deduced to account for the variable stress state and the size effect simultaneously. However, in the evaluation of test results this reference area was used as a constant value although different reference areas resulting from different failure stresses are valid for each data point. Furthermore, the values for the location and shape parameters were obtained fitting directly the raw data to a three-parameter Weibull cdf. Thus, those parameters do not correspond to the uni-axially and uniformly tensioned area. Though the model proposed in Ref. 1 provides an acceptable approximate solution for engineering applications, it is not fully consistent from a statistical point of

* Corresponding author. Tel.: +34 985 18 20 54; fax: +34 985 18 20 55.
E-mail address: afc@uniovi.es (A. Fernández-Canteli).

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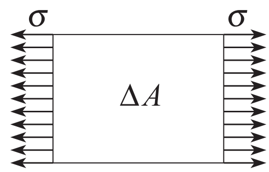

Fig. 1. Uni-axially tensioned area ΔA.

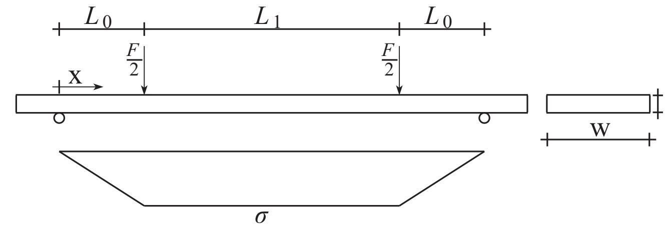

Fig. 2. 4-point bending test.

view, constituting one of the main motivation of the present work.

As this work is mainly concerned with glass, fracture occurring on the side surfaces of the glass beams is not accounted for. One reason is the existence of two different crack populations as the finishing of the side surfaces is different from the one of the lower and upper surfaces of a glass plate. The consideration of surface cracks competing with cracks originated at the edge cuts results in a highly complex problem being statistically recognized as that denoted *confounded data*. Since in the case of ceramics it is correct to consider also the side surfaces, for the sake of completeness we add the extended equations for the corresponding reference areas in the appendix (Eqs. (A.1) and (A.2)).

Frequently, simplifications are made when using the Weibull cdf as it can be verified in the literature[2,3] and design codes[4], where the simpler two-parameter Weibull distribution is used. This cdf is easier to handle because it obviates the consideration of a threshold stress, but the obtained failure predictions may be too conservative[5]. In the case of 4-point bending tests other authors[6] find it sufficiently accurate to use only the central uniformly tensioned lower surface as reference area without considering the two zones adjacent to the beam supports subjected to a linearly increasing stress state. Although this should result in a conservative cdf, it seems necessary to take the whole lower surface of the beams into account because large cracks in the zones adjacent to the supports exposed to stress levels lower than that in the midspan may become determinant and lead to failure.

2. Probabilistic model and proposed methodology

Let us assume that the cdf describing the failure probability of a uni-axially tensioned area ΔA, as shown in Fig. 1, can be expressed in terms of the uni-axially acting stress σ in form of a three-parameter Weibull distribution function $F_{\Delta A}(\sigma)$ (see Weibull[7])

$$F_{\Delta A}(\sigma) = 1 - \exp\left\{-\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right\}; \quad \sigma \geq \lambda, \tag{1}$$

where λ is the location parameter or threshold stress below which no fracture occurs, δ the scale parameter and β the shape parameter. This distribution is theoretically justified by the weakest link principle.

If Expr. (1) is used to calculate the failure probability of a glass beam under 4-point bending (see Fig. 2) it should be noticed that fracture normally starts on the tensioned lower surface of a beam. Accordingly, the surface under tension of the beam is divided into cells having each an equal area of ΔA_i.

In this case, the size effect must be taken into account. Denoting $P_{s,\Delta A}$ the probability of survival for an area ΔA, the probability of survival $P_{s,i}$ for an area $\Delta A_i = n \cdot \Delta A$ ($n \in \mathbb{R}^+$) becomes

$$\begin{aligned} P_{s,i} &= \left[P_{s,\Delta A}\right]^n = [1 - F_{\Delta A}(\sigma)]^{\frac{\Delta A_i}{\Delta A}} \\ &= \exp\left\{-\frac{\Delta A_i}{\Delta A}\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right\}; \quad \sigma \geq \lambda. \end{aligned} \tag{2}$$

Assuming independence between the cells, the failure probability, $P_{f,beam}$, of the whole beam is calculated as follows:

$$P_{f,beam}(\sigma) = 1 - \prod_i P_{s,i}. \tag{3}$$

This can be rewritten as

$$\begin{aligned} P_{f,beam}(\sigma) &= 1 - \exp\left(\log \prod_i P_{s,i}\right) \\ &= 1 - \exp\left(\sum_i \log P_{s,i}\right) \\ &= 1 - \exp\left(\sum_i \log\left[\exp\left\{-\frac{\Delta A_i}{\Delta A}\left(\frac{\sigma(x)-\lambda}{\delta}\right)^{\beta}\right\}\right]\right) \\ &= 1 - \exp\left(\sum_i \left\{-\frac{\Delta A_i}{\Delta A}\left(\frac{\sigma(x)-\lambda}{\delta}\right)^{\beta}\right\}\right). \end{aligned} \tag{4}$$

As the tension along the width w of the beam is assumed to be constant, the area of one cell i is expressed as $\Delta A_i = dx \cdot w$. Now the summation in Eq. (4) can be extended as an integral. As in the three-parameter Weibull distribution the probability of failure for a tension $\sigma(x) < \lambda$ is zero, the lower integration bound for the lateral areas becomes $L_0 \cdot \lambda/\sigma$, where σ is the maximum stress acting on the lower surface of the beam (see Fig. 2). Thus, we get

$$\begin{aligned} P_{f,beam}(\sigma) &= 1-\exp\left[2\cdot\int\limits_{L_0\cdot\lambda/\sigma}^{L_0}\left\{-\frac{dx\cdot w}{\Delta A}\left(\frac{\frac{\sigma}{L_0}\cdot x-\lambda}{\delta}\right)^{\beta}\right\}+\int\limits_{L_0}^{L_0+L_1}\left\{-\frac{dx\cdot w}{\Delta A}\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right\}\right] \\ &= 1-\exp\left[-\frac{w}{\Delta A}\left(2\cdot\left|\frac{\left(\frac{\sigma}{L_0}\cdot x-\lambda\right)^{\beta+1}}{\delta^{\beta}\cdot\frac{\sigma}{L_0}\cdot(\beta+1)}\right|_{L_0\cdot\lambda/\sigma}^{L_0}+L_1\cdot\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right)\right] \\ &= 1-\exp\left[-\frac{w}{\Delta A}\left(\frac{2\cdot L_0\cdot(\sigma-\lambda)}{\sigma\cdot(\beta+1)}+L_1\right)\cdot\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right]. \end{aligned} \quad (5)$$

For 3-point bending, L_1 vanishes and the equation reduces to

$$P_{f,beam}(\sigma)=1-\exp\left[-\frac{w}{\Delta A}\left(\frac{2\cdot L_0\cdot(\sigma-\lambda)}{\sigma\cdot(\beta+1)}\right)\cdot\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right]. \quad (6)$$

If the parameters λ, β and δ were known those expressions would allow predicting the failure probability of a beam under 4-point bending or 3-point bending, respectively. In general, the problem arises the other way round. Data results of bending tests in the form of failure stresses exist and the aim consists in obtaining an analytical expression to describe the material strength for the general case of a uniformly tensioned area ΔA. Based on the independence assumption and the weakest link principle, the cdf can be analytically derived from this analytical expression for any other elementary area. Since the results of the bending tests are referred to a uni-axial stress varying along the length of the beam, we need to refer these results to a reference area A_{ref} subjected to the constant stress σ acting on the central part of the beam having the same failure probability as the entire beam. Replacing ΔA_i by A_{ref} in formula (2) the probability of failure for this area A_{ref} is given by the following expression:

$$P_{f,A_{ref}}(\sigma)=1-\exp\left[-\frac{A_{ref}}{\Delta A}\left(\frac{\sigma-\lambda}{\delta}\right)^{\beta}\right]. \quad (7)$$

Equaling expressions (5) and (7), the area A_{ref} results in

$$A_{ref}=w\cdot\left[\frac{2\cdot L_0}{(\beta+1)}\cdot\left(1-\frac{\lambda}{\sigma}\right)+L_1\right] \quad (8)$$

or

$$A_{ref}=w\cdot\frac{2\cdot L_0}{(\beta+1)}\cdot\left(1-\frac{\lambda}{\sigma}\right) \quad (9)$$

for 4-point and 3-point bending, respectively, whereas expression (8) is equal to the expression obtained in Ref. 1. Note that due to the dependence of A_{ref} on σ, expressions (5)–(7) do not correspond anymore to three-parameter Weibull distribution functions. Expressions accounting also for the tensioned side surfaces of the beam are given in the appendix (Eqs. (A.1) and (A.2)). The equations to calculate the effective surface (herein denoting reference area) in the WeibPar programme[8], which provides the material parameter evaluation for possible use in the CARES[9] reliability calculation, and Ref. 10 coincide with Eqs. (A.1) and (A.2) in case of the two-parameter Weibull distribution ($\lambda = 0$). This confirms the generality of the approach.

In the case of experiments, any failure stress $\sigma_{f,k}$ of the sorted sampled data is assigned to an accumulated failure probability given by[11]

$$P_{f,k,beam}=\frac{k-0.3}{n+0.4}, \quad (10)$$

where n is the sample size and k is the kth element.

Using Eq. (2) we identify any of these accumulated failure probabilities, $P_{f,k,beam}$, with the failure probabilities, $P_{f,k,\Delta A}$, resulting for an area ΔA under constant stress, whereas the size of ΔA, e.g. the size of the finite elements being used in further structural analysis, is arbitrary but unlike A_{ref} its value is constant for all stress levels

$$P_{f,k,\Delta A}=1-(1-P_{f,k,beam})^{\Delta A/A_{ref,k}}. \quad (11)$$

The correctness of Eq. (11) can be easily stated if $P_{f,k,beam}$ in Eq. (11) is substituted by the analytical expression (5), and $A_{ref,k}$ by Eq. (8) that proves $P_{f,k,\Delta A}$ being equal to the searched distribution function (1). As $A_{ref,k}$ depends on the unknown parameters λ and β these have to be estimated. A first approximation is obtained adjusting the experimental data ($P_{f,k,beam}$ versus $\sigma_{f,k}$) to a three-parameter Weibull distribution. This is accomplished by linear regression plotting $\ln(-\ln(1-P_f))$ versus $\ln(\sigma-\lambda)$ and fitting the points to a straight line with slope β and intercept $-\beta\cdot\ln(\delta)$ [11], whereas λ is found by maximizing R^2. Subsequently, a reference area $A_{ref,k}(\sigma)$ is calculated for each data point and the failure probability $P_{f,k,\Delta A}$ valid for ΔA is obtained for each tension $\sigma_{f,k}$ using Eq. (11). Again linear regression is applied to fit these shifted data to a three-parameter Weibull distribution function resulting now in the searched function (1).

The final value of β referred to ΔA should be smaller than that from the initial fitting of the experimental raw data. This will normally be accompanied by a change in the threshold parameter λ. Note that the initially assumed λ, corresponds to a fictitious Weibull distribution that fits approximately the experi-

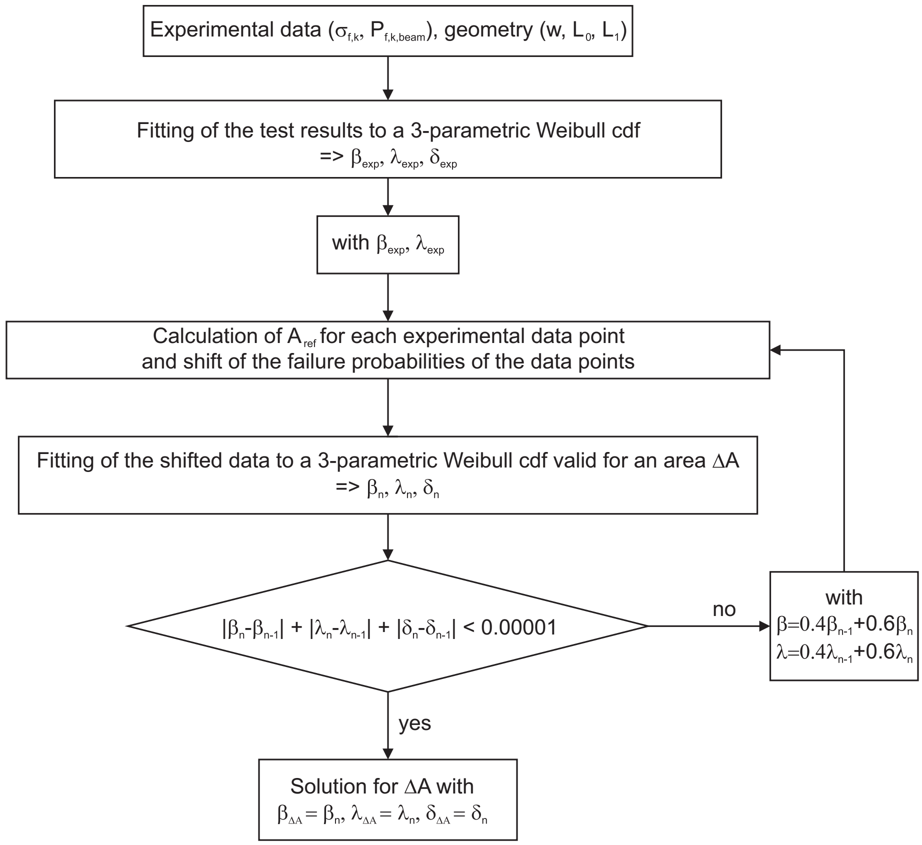

Fig. 3. Flow diagram of method (β_n, λ_n: parameters obtained in actual iteration step; β_{n-1}, λ_{n-1}: parameters obtained in foregoing iteration step).

mental raw data. In this way, the iteration procedure starts using a spurious parameter λ allowing to get the real Weibull primary distribution parameters (among them the true parameter λ) after convergence. Accordingly, the true parameter λ does not change with the scale effect.

To achieve the intended precision, an iteration process is repeated until the β- and λ-values used to calculate A_{ref} become equal to those obtained by linear regression of the data points referred to ΔA. For illustration, a flow diagram of the method applied is shown in Fig. 3. In this paper the linear regression method is used instead of the maximum-likelihood because for a three-parameter Weibull distribution, the latter is known to have problems in estimating the location parameter. More precisely, this leads on one hand to infinite values of the likelihood, and on the other hand to lack of regularity conditions, so that especial and complicated methods should be used, see Castillo and Hadi[12,13].

It has to be mentioned that the iteration process is not necessary when the simplified two-parameter Weibull cdf with $\lambda = 0$ is used for evaluating bending tests, because in this case A_{ref} does not depend on the stress level but only on the geometry of the test specimen and the scale parameter β.

3. Application to test results

For validation of the method proposed we apply it to simulated data of 3- and 4-point bending tests and to real results of 4-point bending tests of monolithic glass beams. Furthermore, the method proposed herein is compared with a least squares best fit method reported by Gross[14] evaluating in both cases the same failure data of 3- and 4-point bending tests on silicon carbide and silicon nitride, respectively. For the iteration procedure, Matlab and the fitting proposed in Ref. 15 are used.

3.1. Application to simulated data

3.1.1. 4-point bending

To simulate 4-point bending tests of beams with dimensions $L_0 = 50$ mm, $L_1 = 150$ mm, $w = 100$ mm, a material with Weibull parameters $\lambda = 42$ MPa, $\beta = 2.6$ and $\delta = 130$ MPa valid for a uniaxial tensioned area $\Delta A = 100\ \text{mm}^2$ has been assumed. Choosing a small value for ΔA in relation to the stressed lower surface of the beam, a good fit of the left-hand tail of the Weibull distribution function (see Fig. 5) is obtained. This is deliberate as in general one is dealing with low failure probabilities in the different practical cases. Furthermore 100 random numbers between 0 and 1 representing fictitious failure probabilities $P_{f,beam}$ are generated. Inserting all these values in Eq. (5) and solving for σ, 100 values of the fracture stress (i.e. the maximum stress in the midspan of the beam which could also provoke fracture at another location with lower stress level) are obtained for a 4-point bending test series. These fracture stresses are treated as failure data by assigning a cumulative failure probability to each of them by means of expression (10). With the proposed

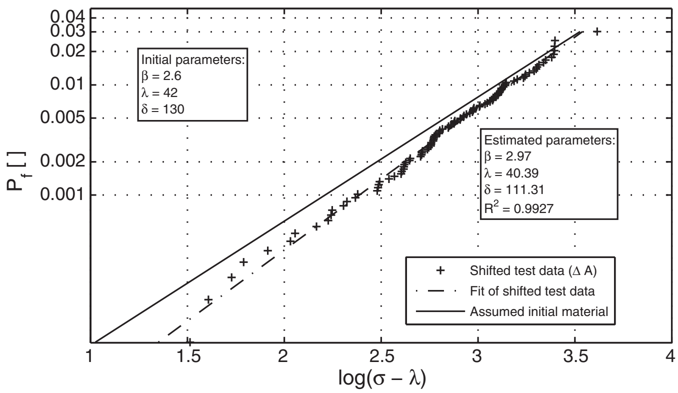

Fig. 4. Regression lines for simulated data of 4-point bending tests.

method we evaluate the simulated experimental data resulting in a Weibull cdf which can be contrasted directly with the cdf assumed initially in the simulation process.

Fig. 4 represents the regression lines for the shifted simulated data points and the assumed initial material referred to an area ΔA in a Weibull probability paper. The cumulative distribution functions for the same simulation are shown in Fig. 5. A zoom on the left-hand tail of the cdfs is depicted in Fig. 6.

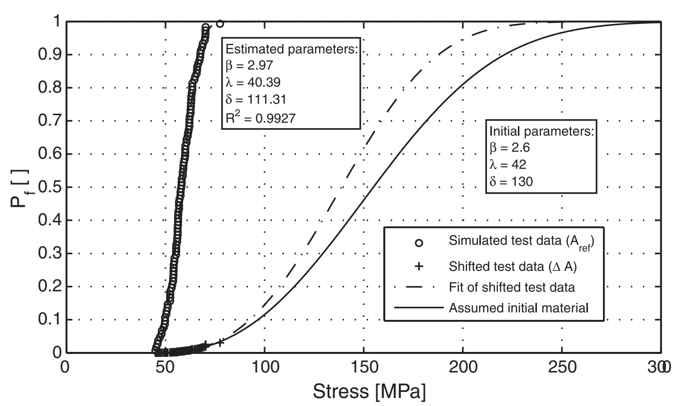

Fig. 5. Cumulative distribution functions for simulated data of 4-point bending tests.

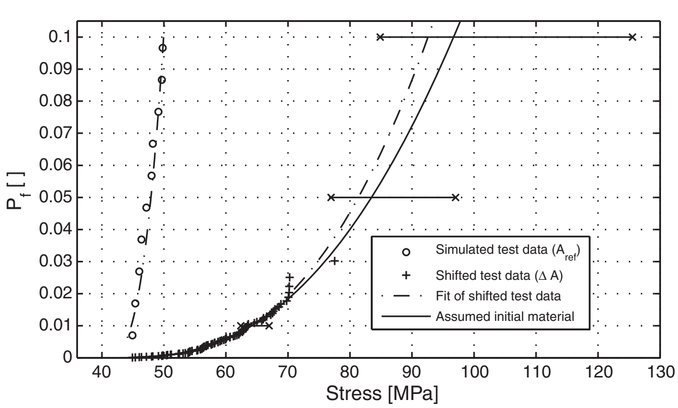

Fig. 6. Left-hand tail of the cumulative distribution functions for 4-point bending tests (the horizontal lines mark the range of values obtained for the percentiles of Table 1).

Table 1
Percentiles 4-point bending.

P_f	1%	5%	10%
Exact value [Mpa]	64.16	83.48	96.71
Mean [Mpa]	64.20	83.79	97.45
Standard deviation [Mpa]	0.76	3.65	6.99
Relative bias [%]	0.06	0.37	0.76
RMSE [MPa]	0.76	3.66	7.03
Maximal value [MPa]	66.96	97.03	125.55
Minimal value [MPa]	62.39	76.94	84.88
Safety coefficient []	0.96	0.86	0.77

The performed simulations allow us to observe that the obtained parameters β, λ and δ do not coincide with those of the initial material used for the simulation of the bending tests. A better verification of the procedure is the comparison among the 1st, 5th and 10th percentiles corresponding to failure probabilities of 1%, 5% and 10%, respectively. By simulation 100 datasets are generated each consisting of 100 values of the fracture stress. For each data set the percentiles are calculated. In Table 1 the mean values and standard deviations for these percentiles are summarized together with the exact percentile-values for the assumed material. The relative bias is calculated as follows:

$$\text{relative bias} = \frac{|\text{exact value} - \text{mean}|}{\text{exact value}}. \tag{12}$$

Additionally, the root mean squared error (RMSE), the maximal and minimal values of the percentiles are presented. The ratio *exact value/maximal value* provides a confident safety coefficient which should be considered when using the determined cdf for engineering calculations. Once the failure stress corresponding to a permitted failure probability is determined, this failure stress has to be reduced by multiplying it by the safety coefficient.

It is obvious that a very good fit for low failure probabilities is achieved, as in the simulations only data points up to a failure probability of about 3% for the cdf referred to an area ΔA are obtained. But also for a failure probability of 10% a quite low relative bias of 0.76% is still attained, though for this failure probability the standard deviation is about 9 times greater than for a failure probability of 1%. Thus, care is advised in front of extrapolation, i.e. using the cdf in regions not covered by data points.

3.1.2. 3-point bending

Another simulation has been carried out for 3-point bending tests. This time, smaller dimensions are used in the simulation: $L_0 = 50$ mm, $w = 30$ mm, $\Delta A = 9$ mm^2. The parameters $\lambda = 42$ MPa, $\beta = 2.6$ and $\delta = 328$ MPa correspond to the same material as in the previous simulation taking into account the size effect by the conversion of the scale parameter δ

$$\delta_2 = \delta_1 \cdot \left(\frac{\Delta A_1}{\Delta A_2}\right)^{1/\beta}. \tag{13}$$

Fig. 7 represents the regression lines for the shifted simulated data points and the assumed initial material referred to an area ΔA in a Weibull probability paper. In Fig. 8 the cumulative dis-

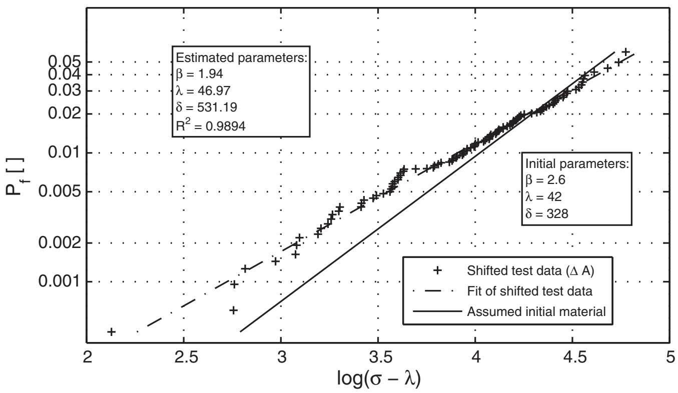

Fig. 7. Regression lines for simulated data of 3-point bending tests.

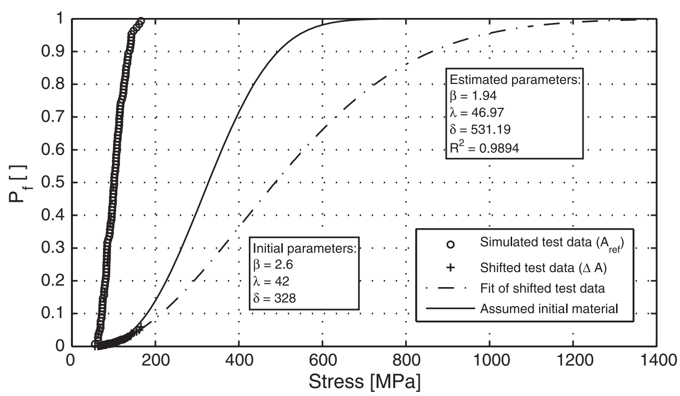

Fig. 8. Cumulative distribution functions for simulated data of 3-point bending tests.

tribution functions for the same simulation are shown. A zoom on the left-hand tail of the cdfs is depicted in Fig. 9.

The evaluation of the 1st, 5th and 10th percentiles results in relative biases of 0.06%, 0.82% and 1.86%, respectively, supplying slightly worse results than in the 4-point bending tests simulation (see Table 2). Due to a higher dispersion of the obtained curves the proposed safety coefficient adopts smaller values than in the 4-point bending tests. On its turn, the RMSE is about 3 times higher than in the 4-point bending test results.

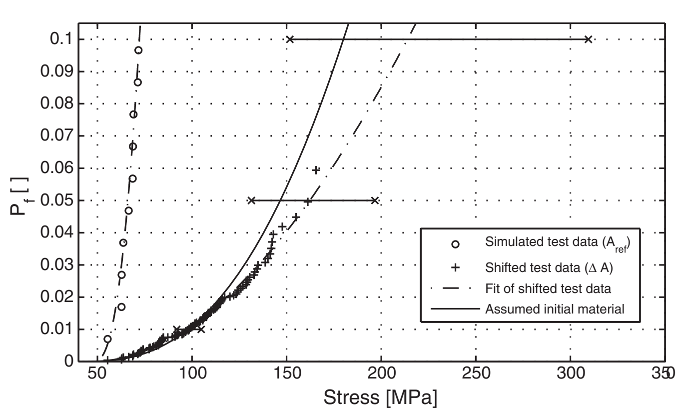

Fig. 9. Left-hand tail of the cumulative distribution functions for 3-point bending tests (the horizontal lines mark the range of values obtained for the percentiles of Table 2).

Table 2
Percentiles 3-point bending.

P_f	1%	5%	10%
Exact value [Mpa]	97.91	146.65	180.03
Mean [Mpa]	97.97	147.86	183.39
Standard deviation [Mpa]	2.14	9.40	21.11
Relative bias [%]	0.06	0.82	1.86
RMSE [MPa]	2.15	9.47	21.38
Maximal value [MPa]	104.85	196.65	309.49
Minimal value [MPa]	91.91	131.43	151.70
Safety coefficient []	0.93	0.75	0.58

3.2. Application to test results

Four-point bending tests on 25 beams of monolithic float glass (tin side under tension) were carried out exhibiting the same dimensions as the ones in the simulation of 4-point bending tests ($L_0 = 50$ mm, $L_1 = 150$ mm, $w = 100$ mm). The test results are listed in Table B.1. Exemplarily, we use a value of 100 mm^2 for ΔA representing a quadratic cell of 1 cm edge length. The data shifted by Eq. (11) fit well to a three-parameter Weibull cdf with parameters $\lambda = 40.94$ MPa, $\beta = 2.80$ and $\delta = 127.3$ MPa.

The resulting scatter $R^2 = 0.9812$ lies in the expected range. Fig. 10 depicts the Weibull probability plot for the shifted data points referred to an area ΔA while Fig. 11 represents the fitted cdf of fracture stress referred to an area ΔA and the data points of failure stress referred to the area A_{ref}.

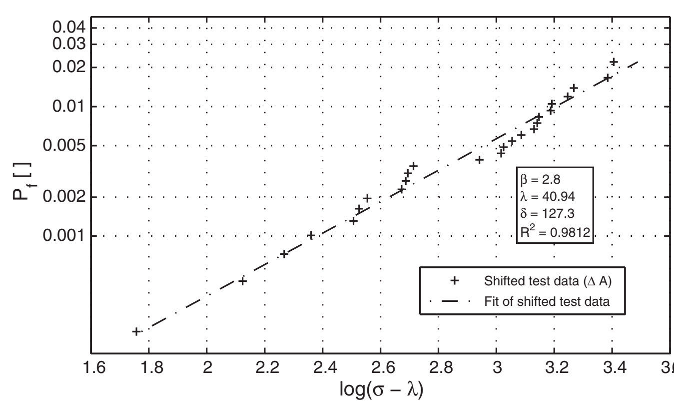

Fig. 10. Regression line for fracture stress of glass beams.

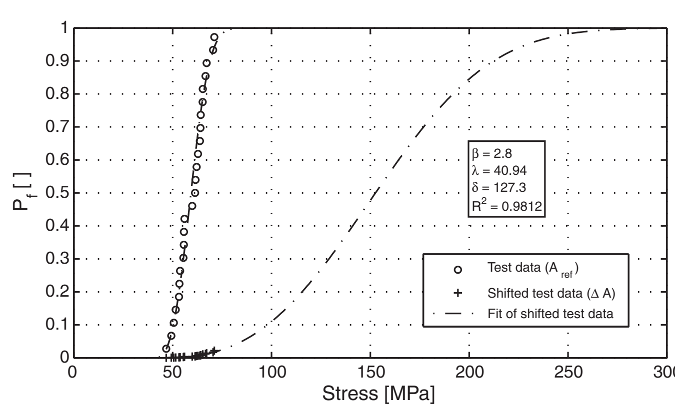

Fig. 11. Cumulative distribution functions for fracture stress of glass beams.

3.3. Comparison with Gross' results

Gross[14] proposed a least squares best fit method to adjust uni-axial failure data to a three-parameter Weibull distribution function. Comparing his procedure with the method proposed his scale parameter δ_{Gross} (with unit MPa m$^{2/\beta}$) can be transformed into our scale parameter δ (with unit MPa) referred to an uniformly tensioned area of ΔA by

$$\delta = \frac{\delta_{Gross}}{\Delta A^{1/\beta}}, \quad (14)$$

which signifies their parameter δ_{Gross} is referred to an area ΔA of 1 m^2. Eq. (14) is equivalent to that used in the WeibPar Manual[8] to relate the Weibull material scale parameter (corresponding to δ_{Gross} referred to unit volume) to the Weibull characteristic strength (corresponding to δ referred to ΔA).

So far we did not consider cracks on the side surfaces of the beams as in glass the cracks on the lower and on the side surfaces belong to different populations (different production steps: lower surface produced by floating and the side surfaces by cutting) leading to the problem of confounded data which is quite involved to account for. In the case of ceramics, all the surfaces of the prismatic specimens, being machined or cut, belong to the same crack population. For that reason, the stressed side surfaces are to be included in the failure calculation. Since the data evaluated by Gross are related to ceramics, the evaluation performed also includes the stressed side surfaces.

As mentioned in Ref. 14, the data of silicon nitride (SNW-1000) are used for four point surface flaw analysis although the fractures occurred due to volume flaws. So the material is not correctly characterized, but a direct comparison with Gross' results is possible.

Using Gross' failure data of silicon nitride and silicon carbide (see Tables I–III of Ref. 14) the proposed method is applied, including the side surfaces in the calculation of A_{ref}, to obtain a cdf valid for an area $\Delta A = 9$ mm^2. To compare both methods, the R^2 of the regression lines in the Weibull paper plot and the sum of residuals squared $\sum_{j=1}^{k}(\sigma_{f,k,comp} - \sigma_{f,k})_j^2$ in the cdf plot are calculated for both fits valid for ΔA, where $\sigma_{f,k,comp}$ is the fracture stress corresponding to the translated probability $P_{f,k,\Delta A}$ computed by the determined cdf for ΔA. To evaluate the sum of residuals squared for the fit made by Gross the test data points are shifted by Eq. (11) using the Weibull parameters λ and β obtained by Gross for the calculation of the reference area and converting his scale parameter to a δ valid for the chosen $\Delta A = 9$ mm^2 by Eq. (14).

For the 4-point bending tests (Figs. 12 and 13) according to R^2, the proposed method seems to provide slightly better results whereas the sum of residuals squared (see Table 3) suggests that the method proposed by Gross is more accurate. This seems logical as the proposed method uses the linear regression method aiming at a maximum of R^2 while Gross uses a least squares best fit method aiming at a minimum of the sum of residuals squared. For the first 3-point bending test data (Table II of Ref. 14) the proposed method gives better results although the difference in the results is quite small. The corresponding regression lines and cdfs are represented in Figs. 14 and 15, respectively. Finally, the

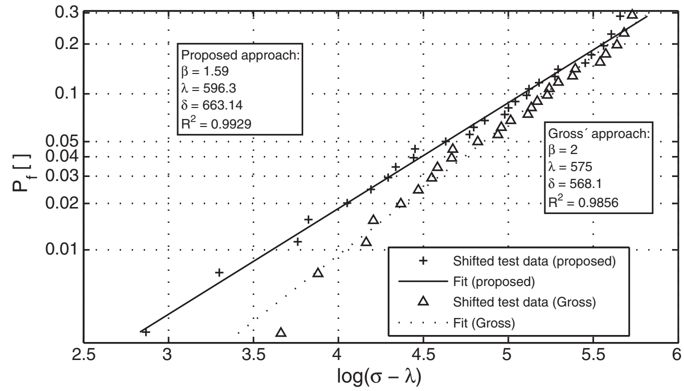

Fig. 12. Regression lines for 4-point bending tests on silicon nitride[14].

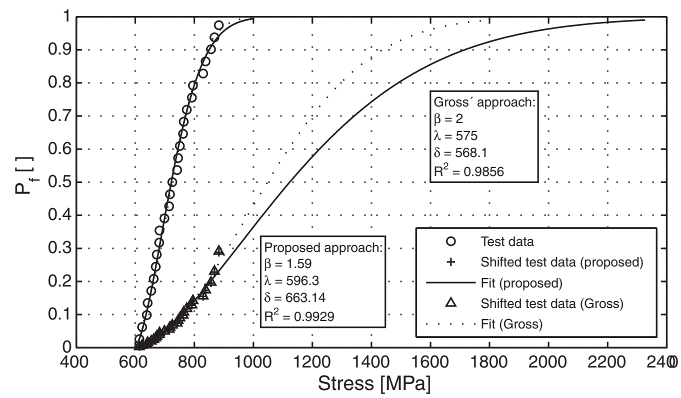

Fig. 13. Cumulative distribution functions for 4-point bending tests on silicon nitride[14].

Table 3
Comparison with parameters obtained by Gross' method.

Test data	Approach	R^2	Sum of residuals squared
4-point bending	Gross	0.9856	2193
(Table I of Ref. 14)	Proposed	0.9929	3627
3-point bending	Gross	0.9663	2478
(Table II of Ref. 14)	Proposed	0.9695	2386
3-point bending	Gross	0.9834	1435
(Table III of Ref. 14)	Proposed	0.9823	1450

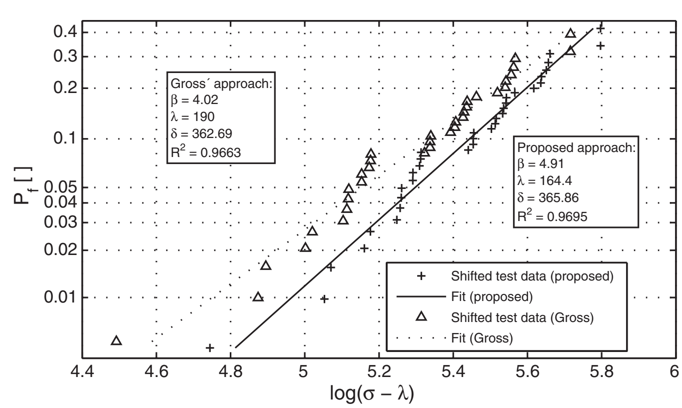

Fig. 14. Regression lines for 3-point bending tests on silicon carbide (transverse annealed)[14].

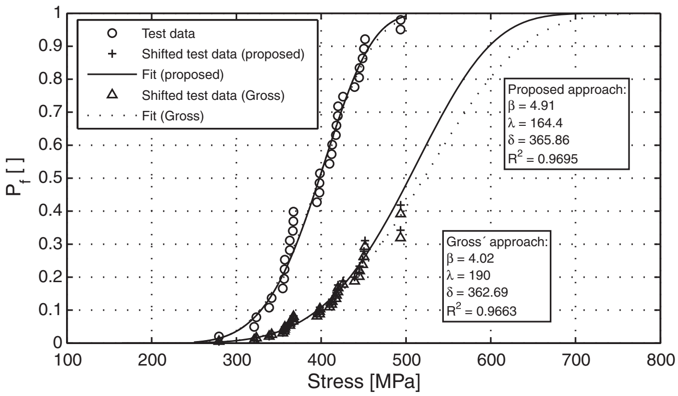

Fig. 15. Cumulative distribution functions for 3-point bending tests on silicon carbide (transverse annealed)[14].

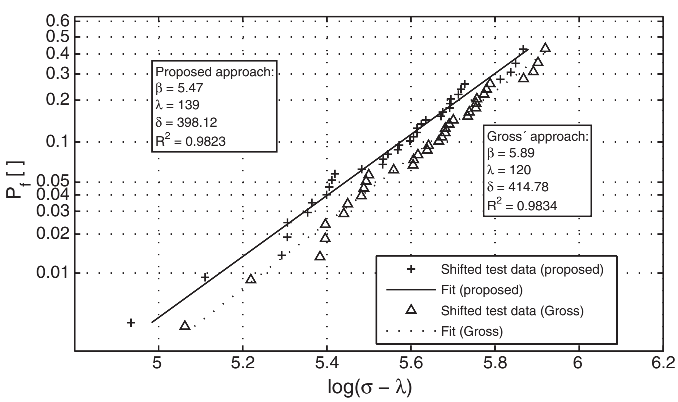

Fig. 16. Regression lines for 3-point bending tests on silicon carbide (longitudinal annealed)[14].

resulting cdfs for the 3-point bending tests (Table III of Ref. 14) are almost coinciding and the values of R^2 and the sum of residuals squared differ only slightly (see Figs. 16 and 17). Thus, it can be concluded that both methods give acceptable results in the evaluation of uni-axial bending tests. An advantage of the method presented here is the easier implementation as the distribution function is reduced to a simple three-parameter Weibull cdf, which parameters can be easily found by linear regression in a Weibull probability plot.

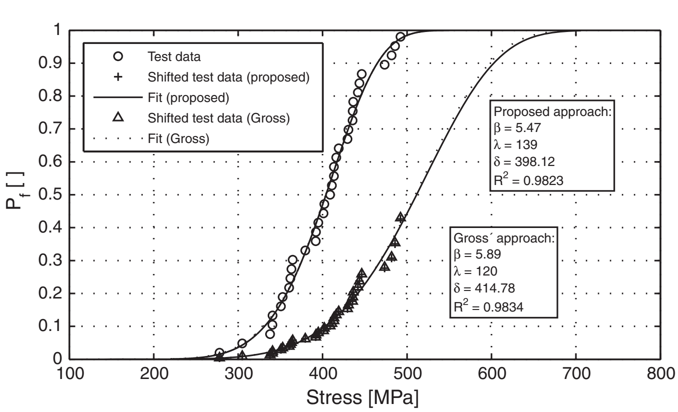

Fig. 17. Cumulative distribution functions for 3-point bending tests on silicon carbide (longitudinal annealed)[14].

Table 4
Geometry for data simulation.

Test arrangement	w [mm]	L_0 [mm]	L_1 [mm]
3-point bending	30	50	–
4-point bending A	100	50	150
4-point bending B	50	30	90

3.4. Extension to the case of different-sized specimens

With the herein proposed method it is also possible to join experimental data from different-sized specimens or test arrangements (e.g. 3- and 4-point bending) to obtain one distribution function describing the material. To demonstrate this, a material with parameters $\beta = 2.6$, $\lambda = 42$ MPa and $\delta = 130$ MPa referred to an area ΔA of 100 mm^2 is assumed for which the fracture data of one 3-point bending and two 4-point bending tests are simulated (dimensions in Table 4). For each test 100 data points are generated and each data series fitted to a three-parametric Weibull-cdf. To adjust the three data series to one distribution function the mean values of λ and β for a first guess are used and fitting is achieved by linear regression method and iteration as pointed out in the flow diagram (see Fig. 3). The resulting regression lines and cdfs are represented in Figs. 18 and 19, respectively.

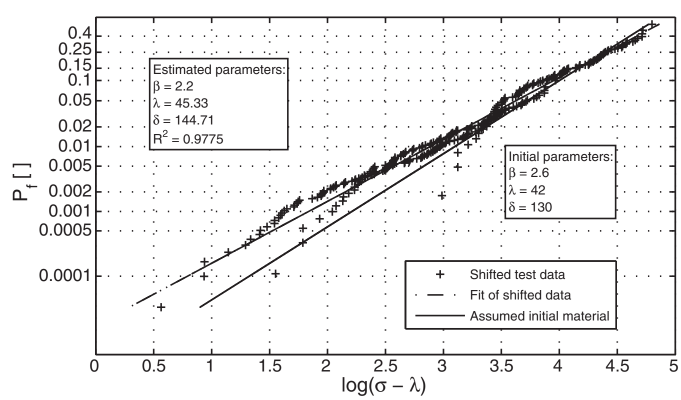

Fig. 18. Three data series fitted to one cumulative distribution function - linear regression.

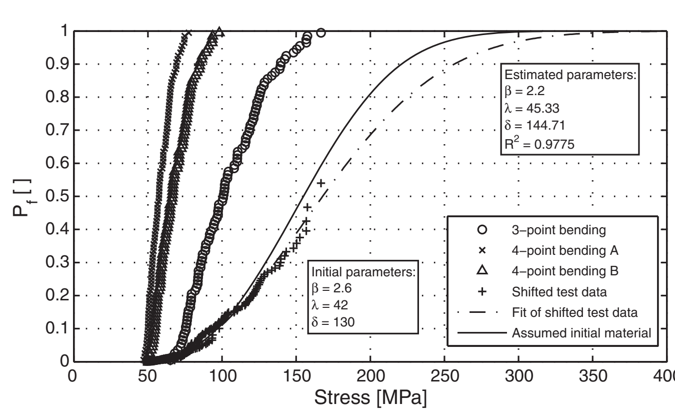

Fig. 19. Three data series fitted to one cumulative distribution function.

4. Conclusions

The herein proposed method enables us to deduce a cdf characterizing the material strength of a uniformly and uniaxially tensioned surface element from bending tests directly. No simplification is adopted concerning the reference area, since considering only the central area as reference area when evaluating 4-point bending tests represents an approximation. Indeed, the influence of the lateral areas and that of the varying side stresses decrease when those are small in relation to the central area. But since large cracks in the lateral areas subjected to lower stress levels than in the midspan may lead to failure, the whole lower surface of the beam should be included in the failure calculation. What is more, for small specimens often the 3-point bending test is employed and this obliges to consider the variant stress state.

In this work only surface defects are considered, though the method is also applicable to materials with volume flaws by substituting the expressions for the reference area by expressions for a reference volume. Changing the expression for the reference area one can evaluate also other test arrangements than 3- or 4-point bending tests.

The proposed procedure can also be applied to multiaxial strength data, e.g. those obtained from ring on ring tests (see ASTM C1499[16]). In such a case, the expression for the reference area must be determined using a multiaxial stress model, e.g. Batdorf or PIA (principle of independent action) model[17]. However, the numerical effort to solve the problem will increase. Salem and Powers[18], dealing with ring on ring tests, provide an approximate solution of the effective surface for the simplified two-parameter Weibull distribution making use of the PIA model.

It is not apparent whether the proposed approach is always conservative or not, as the fits obtained in the simulations predict higher or lower failure probabilities as the exact cdf of the initial material. For this reason, the use of safety coefficients is suggested when resorting to the determined cdfs in engineering design.

Although the whole cumulative distribution function valid for an area ΔA is deduced, only the left-hand tail in probabilistic calculations is considered, as engineering cases are mostly concerned with failure probabilities below 5%. Extrapolation, i.e. use of the cdf in regions not covered by data points must be performed with care because, as can be seen in the diagrams, the deviation of the obtained fits from the initial cdf can adopt quite high values.

In comparison to Gross' least squares best fit method similar results are obtained using the herein proposed method, whereas the latter is easier to implement as the distribution function is reduced to a simple three-parameter Weibull cdf whose parameters can be easily found by linear regression in a Weibull probability plot. Another advantage of the proposed method is that the deduced three-parameter Weibull cdf is directly applicable in finite element calculations or by choosing in the test evaluation an area ΔA equal to the one to be applied in finite element calculations or by transforming the determined scale parameter δ by Eq. (13).

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Appendix A. Reference area in case of consideration of the side surfaces

4-point bending:

$$A_{ref} = w \cdot \left[\frac{2 \cdot L_0}{(\beta+1)} \cdot \left(1 - \frac{\lambda}{\sigma}\right) + L_1 \right] + \frac{L_1 \cdot t}{(\beta+1)} \cdot \left(1 - \frac{\lambda}{\sigma}\right) + \left(\frac{\sigma}{\sigma - \lambda}\right)^{\beta} \cdot \frac{2 \cdot L_0 \cdot t}{(\beta+1)} \cdot \int_{0.5\lambda t/\sigma}^{0.5t} \frac{1}{y} \left(\frac{2}{t} y - \frac{\lambda}{\sigma}\right)^{(1+\beta)} dy \tag{A.1}$$

3-point bending:

$$A_{ref} = \frac{2 \cdot L_0}{(\beta+1)} \cdot \left[w \cdot \left(1 - \frac{\lambda}{\sigma}\right) + t \cdot \left(\frac{\sigma}{\sigma - \lambda}\right)^{\beta} \cdot \int_{0.5\lambda t/\sigma}^{0.5t} \frac{1}{y} \left(\frac{2}{t} y - \frac{\lambda}{\sigma}\right)^{(1+\beta)} dy \right] \tag{A.2}$$

Appendix B. Failure data of bending tests

Table B.1
4-point bending – monolithic float glass.

Specimen number	Failure load – F [N]	Failure strength – σ [MPa]
1	4103	61.55
2	3735	56.03
3	3709	55.64
4	4091	61.37
5	4272	64.08
6	4282	64.23
7	3436	51.54
8	3373	50.60
9	4443	66.65
10	4352	65.28
11	4255	63.83
12	3587	53.81
13	3563	53.45
14	3993	59.90
15	3716	55.74
16	4345	65.18
17	4189	62.84
18	4739	71.09
19	3547	53.21
20	4697	70.46
21	4480	67.20
22	3116	46.74
23	4144	62.16
24	3287	49.31
25	3695	55.43

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